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Spinor Structure of Space–Time[1]

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The spinor structure on space–time manifold is investigated in the frame of Crumeyrolle's approach. Some of his theorems are simplified. The equivalence of this approach to the Milnor and Lichnerowicz one is shown using topological properties of the group space of $\mathscr{L}_0$. The equivalence of any two spinor structures on simply connected space–time is established.

1. INTRODUCTION

The spinor structure on space–time is usually defined (following Milnor and Lichnerowicz) as a prolongation (Trautman, 1973) of the Lorentz structure $\xi_{\mathscr{L}_0}$ on space–time M to a spinor group $SL(2, C)$. If there exists such prolongation $\xi_{SL(2,C)}$ then we can associate with each point $m \in M$ a spinor space $\Sigma(m)$ given by the fiber over m of the associate bundle $\xi_{SL(2,C)}[\Sigma]$, where Σ is a two-dimensional complex vector space equipped with skew bilinear form $\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. However, we can reverse the question and look for conditions which allow us to attach in a continuous way a two-dimensional spinor space to each point of space–time. This problem is investigated in this article in the most convenient framework of the Clifford-bundle approach developed by Crumeyrolle (1969, 1970, 1971, 1975).

This paper is organized as follows. At first we recall some properties of the group space of Lorentz group, and give basic properties of the Clifford algebra as well as of spinor groups (Clifford group, Pin$Q$, Spin$Q$, Spin$_+ Q$). Then we investigate different spinor spaces which we can build at some point of space–time, and give the necessary and sufficient condition that two orthonormal tetrads define the same half-spinor spaces. Next, taking

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into account topological properties of Lorentz group and covering maps, we come to the definition of the spinor structure given by Milnor (1963) and Lichnerowicz (1968). At the end we establish the equivalence between any two spinor structures on simply connected space–time.

2. THE LORENTZ GROUP

In this section we collect a number of facts concerning the Lorentz group $\mathscr{L} = O(3,1)$. By this group we shall understand the set of linear transformations which leaves invariant a nondegenerate quadratic form Q of signature (3,1) on R^4. The group $\mathscr{L}$ is not connected. Let $\mathscr{L}_0$ be the connected component that contains identity. Then $\mathscr{L}$ is the union of the four connected sheets:

$$\mathscr{L} = \mathscr{L}_0 \cup P\mathscr{L}_0 \cup T\mathscr{L}_0 \cup PT\mathscr{L}_0 \tag{2.1}$$

where P is the parity operation, $P(x^0, \mathbf{x}) = (x^0, -\mathbf{x})$; T is the time reversal, $T(x^0, \mathbf{x}) = (-x^0, \mathbf{x})$; and $P^2 = T^2 = (PT)^2 = 1$, $PT = PT$. There exists the short exact sequence

$$0 \to \mathscr{L}_0 \to \mathscr{L} \to Z_2 \otimes Z_2 \to 0 \tag{2.2}$$

which splits on the right. It means that the Lorentz group $\mathscr{L}$ is a trivial extension of the four-element group $Z_2(T) \otimes Z_2(z)$ by $\mathscr{L}_0$, that is, $\mathscr{L}$ is the semidirect product $\mathscr{L} = \mathscr{L}_0 \times (Z_2 \otimes Z_2)$. (Here z is the homomorphic image in $\mathscr{L}$ of the section $-1 \to z$ of the map $\varphi: \mathscr{L} \to \pm 1$; $\varphi(\Lambda) = \operatorname{Det} \Lambda$ for every $\Lambda \in \mathscr{L}$.)

The group $\mathscr{L}_0$ is the proper orthochronous Lorentz group of automorphisms of $R^{3,1}$ preserving the semiorientations: the spacelike one of $R^{3,0}$ and the timelike one of $R^{0,1}$. We shall denote $\mathscr{L}_0 = SO_+(3,1)$. Apart from $\mathscr{L}_0$ none of the components of (1) form a subgroup of $\mathscr{L}$. But there exist the following different invariant subgroups of $\mathscr{L} = O(3,1)$: the proper Lorentz group $\mathscr{L}_+$ of orientation preserving automorphisms of $R^{3,1}$ denoted by $SO(3,1)$ and the orthochronous Lorentz group $\mathscr{L}^\uparrow$ of time orientation preserving automorphisms of $R^{3,1}$. As above, the group $\mathscr{L}$ can be viewed as the group extension of Z_2 by the subgroups $\mathscr{L}_+$ or $\mathscr{L}^\uparrow$. The group $\mathscr{L}_0$ which we will study is equal to $\mathscr{L}_+ \cap \mathscr{L}^\uparrow$. It acts transitively on the positive sheet of the unit timelike pseudosphere of Minkowski space, which means that $\mathscr{L}_0$ acts transitively on the projective space of timelike lines TL in $R^{3,1}$. The isotropy group of timelike lines is $SO(3)$. Because $SO(3)$ is a closed subgroup of $\mathscr{L}_0$, one can define on the group $\mathscr{L}_0$ the structure of a principal $SO(3)$ bundle with the base space $\mathscr{L}_0/SO(3)$. As we have a homeomorphism of $\mathscr{L}_0/SO(3)$ onto TL, it will be desirable to investigate properties of the projective space of timelike lines. Naturally, we can imbed TL into the projective space RP^3 of lines in R^4 in the following way. Let us take the unit positive definite sphere S^3 in R^4. Now we define the projection of the two representative points of a timelike line onto the two points of the intersection of this line with S^3. As is known (Husemoller, 1966), RP^3 can be represented as a three-dimensional disk K^3 of unit radius with antipodal points identified in the boundary S^2 by means of a homomorphism induced by $\phi: K^3 \to S_+{}^3$ ($S_+{}^3$ denotes the subspace of S^3 defined by $x_4 \geqslant 0$), such that

$$\phi(x_1, x_2, x_3) = \left(\frac{x_1}{r} \sin\frac{\pi r}{2}, \frac{x_2}{r} \sin\frac{\pi r}{2}, \frac{x_3}{r} \sin\frac{\pi r}{2}, \cos\frac{\pi r}{2}\right) \tag{2.3}$$

It is obvious that TL is imbedded as the open disk of radius $\frac{1}{2}$. Hence we have TL homeomorphic to R^3. Therefore, since a principal bundle over a contractible base space is a trivial one, we have that $\mathscr{L}_0$ as a principal $SO(3)$ bundle is homeomorphic to $SO(3) \times R^3$:

$$\mathscr{L}_0 \cong SO(3) \times R^3 \tag{2.4}$$

The group $SO(3)$ is the maximal compact subgroup of $\mathscr{L}_0$ and has the topology of RP^3, as can be easily seen. For if we represent $g \in SO(3)$ as a point x_g on the corresponding axis of rotation with $|x_g|$ equal to the angle of rotation, then we have a homeomorphism of $SO(3)$ to a three-dimensional disk of radius π, with antipodal points identified in the boundary S^2, so we have $SO(3) \cong RP^3$. Thus

$$\mathscr{L}_0 \cong RP^3 \times R^3 \tag{2.5}$$

Apart from the maximal compact subgroup $SO(3)$ of the group $\mathscr{L}_0$ we can select (among other subgroups) the Abelian one-parameter subgroup $\mathscr{A}$ generated by L_{03} and the nilpotent two-dimensional subgroup $\mathscr{N}$ generated by $A_1 = L_{01} - L_{31}$ and $A_2 = L_{02} + L_{23}$. Here (L_{01}, L_{02}, L_{03}) are boosts (i.e., generators of the proper Lorentz transformations), while (L_{23}, L_{31}, L_{12}) are operators of the angular momentum, i.e., generators of $SO(3)$. Now an element g of $\mathscr{L}_0$ can be written as

$$g = e^{-i\varphi L_{12}} e^{-i\theta L_{31}} e^{-i\vartheta L_{12}} e^{isL_{03}} e^{-it(L_{01} - L_{31})} e^{-iu(L_{02} - L_{23})} \tag{2.6}$$

The ranges of the parameters s, t, u are

$$-\infty < s, t, u < \infty \tag{2.7}$$

(which is to be expected by virtue of the above consideration which implies that the group space of $\mathscr{L}_0$ is topologically the product of the group space of $SO(3)$ and the three-dimensional Euclidean space). The notation of (2.6) is reclining on the well-known Iwasawa decomposition of $\mathscr{L}_0$, which allows us to treat the Lorentz group as a product of subgroups introduced earlier: $\mathscr{L}_0 = SO(3)\mathscr{A}\mathscr{N}$.

3. SPINOR GROUPS

It is well known that not all representations of the orthogonal Lie group $\mathscr{L} = O(3,1)$ can be obtained from the vector representation by constructing tensor products and decomposing them onto simple representation spaces. The simplest presentation of representations which cannot be obtained in this way (spinor representations) is based on the theory of Clifford algebras, so we begin with a short exposition of this merit.

3.1. Clifford Algebras. If we have an n-dimensional vector space E over R, equipped with a nondegenerate quadratic form Q of signature $(k, n-k)$ (and a bilinear form $B(x, y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]$ associated to Q) we can construct the Clifford algebra $C(Q)$ corresponding to this form. The underlying vector space of the Clifford algebra $C(Q)$ is isomorphic to that of the exterior algebra $\wedge E$ of E, but the multiplication in $C(Q)$ is specified by the form Q (Chevalley, 1954) in such a way, that for any $x \in E$ the operator of left multiplication by x is $(L_x + \delta_x)$. Here L_x is the operator of left exterior multiplication by x in $\wedge E$ and δ_x is the antiderivation of $\wedge E$ such that $\delta_x y = B(x, y)\cdot 1$ for $y \in E$. Thus the exterior algebra $\wedge E$ may be identified with the Clifford algebra of the zero form on E. We shall now indicate a more abstract construction. Let us take the tensor algebra T of the vector space E. Let J be the ideal generated in T by elements $a \otimes x \otimes a$, where $x \in E$, $\alpha \in T(E)$. The exterior algebra $\wedge E$ is the factor algebra $\wedge E = T/J$ (Geroch, 1968). Now let us consider the ideal I of T generated by elements $x \otimes x - Q(x)\cdot 1$ for all $x \in E$. The Clifford algebra $C(Q)$ of the quadratic form Q is the factor algebra T/I. It can be easily seen that $I = J$ in the case of the zero quadratic form Q (Bourbaki, 1959). The natural mapping of T onto $C(Q) = T/J$ induces an isomorphism of E into $C(Q)$, and henceforth we shall identify the elements of E with their images in $C(Q)$. Thus E will be considered as a subspace of $C(Q)$. We have

$$x^2 = Q(x)\cdot 1 \qquad \text{for } x \in E \tag{3.1}$$

and

$$xy + yx = 2B(x, y)\cdot 1 \qquad \text{for } x, y \in E \tag{3.2}$$

The dimension of the algebra $C(Q)$ is equal to 2^n, and when $(e_1, \ldots, e_n)$ is a base of E then elements $e_{i_1}e_{i_2}\cdots e_{i_k}$ of $C(Q)$ with a strictly increasing sequence of integers $1 \leqslant i_1 < i_2 \cdots < i_k \leqslant n$ form a base of $C(Q)$. Because (Husemoller, 1966) the tensor algebra is Z_2 graded onto tensors of even and odd degree, then so is $C(Q) = C$: $C = C_+ \oplus C_-$ with the subalgebra C_+ being the image in C of the subalgebra $\sum_{k\geq 0} T^{2k}(E)$ of the tensor algebra $T(E)$, and submodule C_- being the same for the submodule $\sum_{k\geq 0} T^{2k+1}(E)$ of T.

Let us introduce for a future use the notion of the main antiautomorphism α of the Clifford algebra C. It is defined (Bourbaki, 1959) in a natural manner by the antiautomorphism α^T of T, which is an extension of all the mappings $\alpha_k\colon x_1 \otimes \cdots \otimes x_k \to x_k \otimes x_{k-1} \otimes \cdots \otimes x_1$. For the main antiautomorphism α we have

$$\begin{aligned} \alpha(x) &= x \qquad x \in E \\ \alpha(C_+) &= C_+ \\ \alpha(C_-) &= C_- \end{aligned} \tag{3.3}$$

Let us limit ourselves to the case of $2r$-dimensional vector spaces E. Then the center Z of C is one dimensional, $Z = R\cdot 1$, and the Clifford algebra C of Q is a central simple algebra. The central fact in the theory of Clifford algebras is that when C is simple then there exists exactly one, up to equivalence, finite-dimensional simple representation ρ of this algebra. It is called the spinor representation of C, and the dimension of ρ equals 2^r.

3.2. Clifford Groups. As we have said, the vector space E can be considered as a subspace of $C(Q)$. Let us take an invertible element $g \in C(Q)$ such that

$$gxg^{-1} \in E \qquad \text{for every } x \in E \tag{3.4}$$

The set of all such elements forms a group G called the Clifford group. The mapping $\varphi\colon G \to \operatorname{Aut} E$ given by (3.4) is a linear representation of G which we shall call the vector representation of G to distinguish it from the spin representation.

As for every $g \in G$ we have by (3.1)

$$Q(gxg^{-1})\cdot 1 = (gxg^{-1})^2 = Q(x)\cdot 1 \tag{3.5}$$

so φ maps G into the orthogonal group $O(Q)$ and in the case of even-dimensional E, $\varphi(G) = O(Q)$ with the kernel isomorphic to the group $GL(1)$ of nonzero real numbers. Now for any nonsingular element x of E (only then x is invertible, $x^{-1} = (Q(x))^{-1}x$) we see from (3.1) and (3.2) that

$$xyx^{-1} = Q(x)^{-1}xyx = -y + \frac{B(x, y)}{Q(x)}\,x \tag{3.6}$$

Hence x belongs to G and $\varphi(x)$ is given by the symmetry with respect to the hyperplane orthogonal to x.

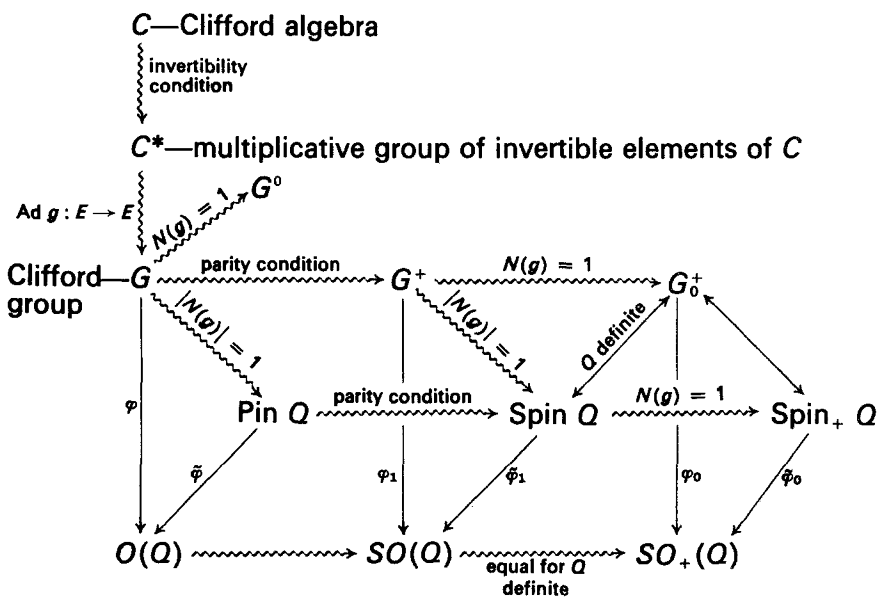
Because E is even dimensional, every element from G may be written as (Kobayashi and Nomizu, 1963)

$$g = \lambda x_1 \cdots x_k \qquad \begin{array}{l} 0 \neq \lambda \in R \\ x_i \text{ nonsingular} \end{array} \tag{3.7}$$

The subset of elements of G with k even form a subgroup G^+ of the Clifford group G, called the special Clifford group, and mapping $\varphi_1 = \varphi|_{G^+}$ maps G^+ onto the special orthogonal group $SO(Q)$. Now we shall take into consideration the main antiautomorphism α of the algebra C introduced earlier. For any element g of the Clifford group G, $\alpha(g)$ belongs to G as well, and $\alpha(g)g \neq 0$, $\alpha(g)g \in R$. Thus we can define the "norm" homomorphism $N: G \rightarrow R$ given by

$$N(g) \cdot 1 = \alpha(g)g \tag{3.8}$$

We shall denote by G_0 the group of elements g of G such that $N(g) = 1$, and by $G_0{}^+$ the group $G_0 \cap G^+ = G^+/GL(1)$, called the reduced Clifford group. Let φ_0 be the restriction of φ to $G_0{}^+$. The subgroup of G isomorphic to the kernel of $|N|$ will be called Pin$Q$ and is in fact the quotient group $G/GL_+(1)$, where $GL_+(1)$ is the subgroup of nonzero positive numbers of $GL(1)$. (See Fig. 1.)



$\mathrm{Ker}\,\varphi = GL(1)$ $\mathrm{Ker}\,\varphi_1 = GL(1)$ $\mathrm{Ker}\,\varphi_0 = \mathrm{Ker}\,\tilde{\varphi}_0 = Z_2$

$\mathrm{Ker}\,\tilde{\varphi} = Z_2$ $\mathrm{Ker}\,\tilde{\varphi}_1 = Z_2$

Fig. 1. Subgroups of the Clifford group.

The mappings $\tilde{\varphi}, \tilde{\varphi}_1, \tilde{\varphi}_0$ are restrictions of $\varphi, \varphi_1, \varphi_0$, respectively. The group Spin$Q$ will be the quotient group $G^+/GL_+(1)$ equal to Pin$Q \cap G^+$. Thus every element g of Pin$Q$ has the form $g = \lambda x_1 \cdots x_k$ with $N(g) = \pm 1$ and every element g' of Spin$Q$ can be written as $g' = \lambda' x_1 \cdots x_k$, with k even and $N(g') = \pm 1$ (λ, λ' are nonzero real numbers). We then have obtained that the element $\tilde{\varphi}_1(g')$ is the product of an even number of symmetries; hence $\tilde{\varphi}_1(g')$ belongs to $SO(Q)$, and similarly $\tilde{\varphi}(g)$ belongs to $O(Q)$. Because the kernel of φ as well as the kernel of φ_1 is one dimensional, the Lie algebras of the groups Pin$Q$, Spin$Q$, Spin$_+Q$ and $O(Q)$, $SO(Q)$, $SO_+(Q)$ are the same. Investigations of these algebras show (Chevalley, 1954; Crumeyrolle, 1971) that Pin$Q$, Spin$Q$, and Spin$_+Q$ are covering groups of $O(Q)$, $SO(Q)$, and $SO_+(Q)$, respectively, with the covering mappings $\tilde{\varphi}$, $\tilde{\varphi}_1$, and $\tilde{\varphi}_0$. Thus we have isomorphisms

$$O(Q) \cong \frac{\mathrm{Pin}Q}{Z_2} \qquad SO(Q) \cong \frac{\mathrm{Spin}Q}{Z_2} \qquad SO_+(Q) \cong \frac{\mathrm{Spin}_+Q}{Z_2}$$

It can also be seen that

$$\frac{\mathrm{Pin}Q}{\mathrm{Spin}Q} \cong \frac{O(Q)}{SO(Q)} \cong Z_2$$

and

$$\frac{\mathrm{Spin}Q}{\mathrm{Spin}_+Q} \cong \frac{SO(Q)}{SO_+(Q)} \cong Z_2$$

if the form Q is indefinite. Thus the group $SO_+(Q)$ is of index 4 in $O(Q)$.

Coming back to the case of four-dimensional vector space $R^{3,1}$ and to the Lorentz group $\mathscr{L} = O(3,1)$, which is of interest to us, we see that $\mathscr{L}$ has four connected components (compare with Section 2) with the identity component equal to $SO_+(3,1) = \mathscr{L}_0$. Moreover, Spin$_+(3,1)$ is the identity component of Pin$(3,1)$ and is mapped onto $SO_+(3,1)$. Because it can be shown (Chevalley, 1954) that $C_+(3,1)$ is isomorphic to $\mathrm{End}_C(C^2)$, we obtain that Spin$_+(3, 1)$ is isomorphic to $SL(2,C)$. Let us recall that the group $SO_+(3,1) = \mathscr{L}_0$ has the topology of $SO(3) \times R^3$. Because Spin$_+(3,1)$ is the simply connected twofold cover of $SO_+(3,1)$, it must be isomorphic to the simply connected twofold cover of $SO(3) \times R^3$; thus we have Spin$_+(3,1) \cong SU(2) \times R^3 \cong S^3 \times R^3$.

4. SPINORS

As we have said, in the case of even-dimensional vector space E the Clifford algebra $C(Q) = C$ is the central simple algebra, and hence has exactly one, up to equivalence, irreducible representation ρ. We call the space of this representation the spinor space of our quadratic form Q of signature $(u - k, k)$, $k \leqslant n/2$. Thus the spinor is an element of $2^{n/2}$-dimensional vector space S (because in the regular representation ρ occurs exactly as many times as its dimension). There is a beautiful method of constructing the space S, which shows that in fact the spinor space is closely related to our original space E. In order to obtain this, let us take the complexifications E_C, Q', and $C(Q') = C'$ of E, Q, and $C(Q)$, respectively. Then every orthogonal frame $\epsilon_0 = \{e_1, \ldots, e_{2r}\}$ in E gives rise to the decomposition of E_C onto two totally singular subspaces N and P, known as the Witt decomposition, where N is spanned by vectors

$$x_1 = \frac{1}{2^{1/2}}(e_1 + e_n),$$

$$x_2 = \frac{1}{2^{1/2}}(e_2 + e_{n-1}), \ldots, \qquad x_k = \frac{1}{2^{1/2}}(e_k + e_{n-k+1}),$$

$$x_{k+1} = \frac{1}{2^{1/2}}(e_{k+1} + ie_{n-k}), \ldots, \qquad x_r = \frac{1}{2^{1/2}}(e_r + ie_{n-r+1}) \tag{4.1}$$

whereas P by vectors

$$y_1 = \frac{1}{2^{1/2}}(e_1 - e_n),$$

$$y_2 = \frac{1}{2^{1/2}}(e_2 - e_{n-1}), \ldots, \qquad y_k = \frac{1}{2^{1/2}}(e_k - e_{n-k+1}),$$

$$y_{k+1} = \frac{1}{2^{1/2}}(e_{k+1} - ie_{n-k}), \ldots, \qquad y_r = \frac{1}{2^{1/2}}(e_r - ie_{n-r+1}) \tag{4.2}$$

Thus $E_C = N \oplus P$, and the elements $(x_1, \ldots, x_r, y_1, \ldots, y_r)$ form a base of E_C which we shall denote by ω_0. Moreover, we have

$$B'(x_i, x_j) = B'(y_i, y_j) = 0, \qquad 2B'(x_i, y_j) = \delta_{ij} \tag{4.3}$$

with B' being the bilinear form associated to Q'. Now, if we denote by f the product of all the base elements of P,

$$f = y_1 \cdots y_r \tag{4.4}$$

we find that $C'f$ is a minimal left ideal of C' and

$$C'f = C^N f \tag{4.5}$$

Here C^N (as well as C^P) is the subalgebra of $C' = C(Q')$ generated by the maximally isotropic subspace N (respectively, P) of E_C. If we take now into account that C' is simple, $C'f$ is a minimal left ideal, and

$$\dim C'f = \dim C^N f = \dim \wedge N = 2^r \tag{4.6}$$

then we obtain the spinor representation ρ' of C' on $C^N \cong \wedge N$, given by

$$(\rho'(v)u)f = vuf, \qquad v \in C' \qquad u \in \wedge N \tag{4.7}$$

and equivalent to it spinor representation ρ on $C'f = C^N f$, given by

$$\rho(v)(wf) = vwf \qquad v, w \in C' \tag{4.8}$$

[The representation of $\mathrm{Pin}\,Q$ induced by ρ is irreducible, because $C(Q)$ can be obtained by means of linear combinations of elements belonging to $\mathrm{Pin}\,Q$ and because $C(Q')$ is the complexification of the Clifford algebra $C(Q)$.] So the spinor will be an element of 2^r-dimensional vector space $S = C'f$ on which acts the irreducible representation of the group $\mathrm{Pin}\,Q$. We see that the set of elements $x_{i_1}\cdots x_{i_k}y_1\cdots y_r$ with $1 \leqslant i_1 < i_2 \cdots < i_k \leqslant r$ forms a base $\mathscr{S}_0$ for the spinor space S. Now, when we take a product of $\gamma \in \mathrm{Pin}\,Q \subset C(Q')$ by an element of $\mathscr{S}_0$, we obtain again an element of the chosen minimal left ideal, and $\gamma\mathscr{S}_0$ will be some other base for the spinor space $S = C'f$:

$$\mathscr{S} = \gamma\mathscr{S}_0 = \{\gamma x_{i_1}\cdots x_{i_k}y_1\cdots y_r\} = \{x'_{i_1}\cdots x'_{i_k}\gamma y_1\cdots y_r\}$$
$$1 \leqslant i'_1 < i'_2 \cdots < i'_k \leqslant r \tag{4.9}$$

Here $x'_{i_n} = \gamma x_{i_n}\gamma^{-1} = \tilde{\varphi}(\gamma)x_{i_n}$, so we have that elements $x'_1, \ldots, x'_r$ are related with another Witt decomposition of E_C given by orthogonal base ϵ in E, $\epsilon = \epsilon_0\tilde{\varphi}(\gamma)$. Because every spinor frame of the spinor space S can be obtained as $\mathscr{S} = \gamma\mathscr{S}_0$ with $\gamma \in \mathrm{Pin}\,Q$, then from (4.9) we can get $\mathscr{S}$ as uniquely defined by (ϵ, γ), where $\epsilon = \epsilon_0\varphi(\gamma)$. We see that the base ϵ alone defines $\mathscr{S}$ only up to the sign. Let us take two spinor bases $\mathscr{S} \equiv (\epsilon, \gamma)$ and $\tilde{\mathscr{S}} \equiv (\tilde{\varepsilon}, \tilde{\gamma})$. Then

$$\tilde{\mathscr{S}} = \tilde{\gamma}\mathscr{S}_0 = \sigma\gamma\mathscr{S}_0 = \sigma\mathscr{S} \tag{4.10}$$
$$\sigma, \gamma, \tilde{\gamma} \in \mathrm{Pin}\,Q \qquad \sigma = \tilde{\gamma}\gamma^{-1}$$

Now it can be easily seen that if the spinor $s \in S$ has in the base $\mathscr{S} \equiv (\epsilon, \gamma)$ components

$$s = s^{i_1\cdots i_k}x'_{i_1}\cdots x'_{i_k}\gamma y_1\cdots y_r \tag{4.11}$$

then in the base $\tilde{\mathscr{S}} \equiv (\tilde{\varepsilon}, \tilde{\gamma})$ it will be given by

$$s = \tilde{s}^{i_1\cdots i_k}\tilde{x}'_{i_1}\cdots \tilde{x}'_{i_k}\tilde{\gamma}y_1\cdots y_r = \rho(\sigma^{-1})s^{i_1\cdots i_k}\tilde{x}'_{i_1}\cdots \tilde{x}'_{i_k}\tilde{\gamma}y_1\cdots y_r \tag{4.12}$$

We want to point out that the spinor space is specified as the minimal left ideal $C'f$ of $C(Q')$ defined by some fixed totally isotropic subspace P of E_C. In our consideration the subspace P had been obtained from a fixed orthogonal ϵ_0 in E, and isotropic r vector f was the product of the elements of the base of P, given by ϵ_0 according to (4.2). It is very useful to know that in order that f and f' define the same minimal left ideal (that is $C'f = C'f'$), it is necessary and sufficient that f and f' define the same maximal totally isotropic subspace of E_C. Namely, it is obvious that when $f' = \lambda f$, then $Cf' = Cf$. Conversely, let $Cf' = Cf$. Because $f' = \gamma f \gamma^{-1}$ (Chevalley, 1954, pp. 16 and 72) with $\gamma \in \operatorname{Pin} Q'$, we have $Cf\gamma^{-1} = Cf$. But $Cf\gamma^{-1}$ and Cf are minimal left ideals, and it follows that the element $f\gamma^{-1}$ must be given by sf for some $s \in C(Q')$. On the other hand, sf is an element of the minimal left ideal $C'f$ as well as $f\gamma^{-1}$ is an element of the minimal right ideal. But it is well known (Chevalley, 1954) that the intersection of any minimal right ideal of the Clifford algebra with any minimal left ideal is a one-dimensional vector space. In our case this intersection is

$$sf = f\gamma^{-1} = \mu f \qquad \text{with } \mu \in Z \tag{4.13}$$

Now, when we apply the main antiautomorphism α to the expression (4.13), we obtain

$$\alpha(f\gamma^{-1}) = \alpha(\gamma^{-1})\gamma^{-1}\gamma\alpha(f) = \mu\alpha(f) \tag{4.14}$$

and because $\alpha(\gamma^{-1})\gamma^{-1} = N(\gamma^{-1})$ and $\alpha(f) = (-1)^{r(r-1)/2}f$, we have

$$\gamma f = N(\gamma)\mu \tag{4.15}$$

Thus we arrive at the conclusion that

$$f' = \gamma f \gamma^{-1} = N(\gamma)\mu^2 f = \lambda f \tag{4.16}$$

This result tells us that two orthogonal bases ϵ and ϵ' in E define the same spinor spaces if and only if the Witt bases ω and ω' related with ϵ and ϵ', respectively, by (4.1) and (4.2) define the identical maximally isotropic subspaces $P' = P$, that is, $f' = \lambda f$ with $\lambda \neq 0$. As we know, the orthogonal bases ϵ of E are permuted transitively among themselves by the operation $O(Q)$. The covering mapping $\tilde{\varphi}$ allows us to put $f' = \gamma f \gamma^{-1}$ with $\gamma \in \operatorname{Pin} Q$ and $O(Q) \ni \tilde{\varphi}(\gamma)\colon \epsilon \to \epsilon'$. (Naturally we have ambiguity of the sign of γ which cannot be avoided.) It appears, however, that even though frames ϵ and ϵ' do define the same spinor space, i.e., $\gamma f \gamma^{-1} = \lambda f$, then the related spinor frames $\mathscr{S}_0$ and $\mathscr{S}'_0$ do not have to be $\operatorname{Pin} Q$ equivalent. We show now that $\mathscr{S}_0$ and $\mathscr{S}'_0$ are connected by an element of $\operatorname{Pin} Q$ only when $\lambda = \pm 1$. For this we shall consider the spinor bases $\mathscr{S}_0$ and $\mathscr{S}'_0$ given by ω and ω', respectively:

$$\mathscr{S}_0 = \{x_{i_1} \cdots x_{i_k} f\} \qquad \text{and} \qquad \mathscr{S}'_0 = \{x'_{i_1} \cdots x'_{i_k} f\} \tag{4.17}$$

Now we have from (4.13) and (4.16)

$$\mathscr{S}'_0 = \{\gamma x_{i_1} \cdots x_{i_k} \gamma^{-1} \gamma f \gamma^{-1}\} = \{\gamma x_{i_1} \cdots x_{ik} \mu f\} = \mu\gamma\mathscr{S}_0 \tag{4.18}$$

So, if $\mu \neq \pm 1$ we obtain that the spinor frames $\mathscr{S}_0$ and $\mathscr{S}'_0$ defined by ϵ and ϵ' [which are $O(Q)$-equivalent] are G equivalent but not $\operatorname{Pin} Q$ equivalent [$\operatorname{Pin} Q \cong G/GL_+(1)$]. Henceforth if we want to get the same spinor space by maximally r-isotropic vectors f and f' (given by ϵ and ϵ') provided with Pin$Q$-equivalent spinor bases, then f' must be

$$f' = \gamma f \gamma^{-1} = \pm f \tag{4.19}$$

The elements γ of the group Pin$Q$, which satisfy the condition of (4.19), form the subgroup of Pin$Q$ which we shall denote by $\mathscr{H}$. Now when we require that the orthogonal bases ϵ and ϵ' of E define the same spinor space and equivalent spinor bases, then ϵ and ϵ' have to be obtained one from another by means of the operation $g \in O(Q)$ such that $\tilde{\varphi}^{-1}(g) \subset \mathscr{H}$. If we denote by H the group $H = \tilde{\varphi}(\mathscr{H}) \subset O(Q)$, we have that g must be an element of H. When we start with some orthogonal frame ϵ' of E which is linked to ϵ_0 by some element $g \in O(Q)$, $g \notin H$, then ϵ' will define another spinor space which will be the space of an equivalent irreducible representation of the group Pin$Q$. If we restrict ourselves to the even elements of the group Pin$Q$, that is, to the group Spin$Q$, then the spinor space $S = C'f \equiv \wedge N$ splits into two subspaces S_e and S_0 invariant under the Spin$Q$ group. It is easy to see this when we take into account that for every $x \in E_C = N \oplus P$, $x = (y, z)$ with $y \in N$, $z \in P$, the spinor representation $\rho'(x) = \rho'(y, z)$ is the operation of left multiplication by y in $C^N \equiv \wedge N$ and the homogeneous antiderivation of degree -1 of C^N. Thus $\rho'(x)u = \rho'(y, z)u = y \wedge u + 2B(z, u)$ for $u \in N$. It follows immediately that every operation $\rho'(v)$ with $v \in C_+(Q) \subset C'_+(Q)$ transforms every one of the half-spinor spaces $C_+{}^N = C^N \cap C_+ \equiv C_+{}^N f$ and $C_-{}^N = C^N \cap C_- \equiv C_-{}^N f$ into itself. But $S = C'f \equiv C^N$, and ρ is equivalent to ρ', then we deduce that the spinor representation ρ of Spin$Q$ is the sum of two inequivalent irreducible representations on the spaces S_e and S_0. The elements of these 2^{r-1}-dimensional spaces are called half-spinors. We shall return to the half-spinors later on, but now we want to consider the tensor product of the space S of spinors with itself.

It can be proved that there exists an isomorphism φ of the space $S \otimes S$ onto $C(Q')$ given by

$$\varphi(u \otimes v) = \tilde{u} f \alpha(\tilde{v}) \tag{4.20}$$

where, $u, v \in S = C'f$, and $\tilde{u}, \tilde{v} \in C^N \equiv S$ (α is the main antiautomorphism introduced earlier). Of course the tensor product $S \otimes S$ is the space of the tensor representation of the group Pin$Q$, which naturally is a reducible representation. Now let us note that

$$\varphi(\rho(s)u \otimes \rho(s)v) = s\tilde{u}f\alpha(\tilde{v})\alpha(s) = N(s)s\varphi(u \otimes v)s^{-1} \tag{4.21}$$

Thus if we restrict ourselves to the case of the group G_0 of elements s with norm equal to 1, and if we recall that the vector space E is identical with some subspace of the Clifford algebra $C(Q) \subset C(Q') \equiv S \otimes S$, then we can see from (4.21) that to every transformation $\rho(s)$ of $S \otimes S$ there corresponds a transformation of E belonging to $O(Q)$. Nevertheless this situation is not quite satisfactory, because we prefer to have as the basic quantities the half-spinors which play the fundamental role in the physical theories. It is clear from the previous considerations that to get this we have to take only the even elements of G_0. Thus we should take the group $G_0{}^+ = \mathrm{Spin}_+ Q$ as the symmetry group of the spinor space. It is evident from Figure 1 that the symmetry group of our original vector space E is the group $SO_+(Q)$.

In the case we are dealing with, the space E is a four-dimensional vector space over R equipped with a quadratic form $Q(3,1)$. Let us take the orthogonal base $\epsilon_0 = \{e_1, e_2, e_3, e_4\}$ with

$$Q(e_i) = \begin{cases} +1 & i = 1, 2, 3 \\ -1 & i = 4 \end{cases} \tag{4.22}$$

The Witt base ω_0 of the complexification E_c of the space E, corresponding to ϵ_0, is given by

$$x_1 = \frac{1}{2^{1/2}}(e_1 + e_4) \qquad y_1 = \frac{1}{2^{1/2}}(e_1 - e_4)$$

$$x_2 = \frac{1}{2^{1/2}}(e_2 + ie_3) \qquad y_2 = \frac{1}{2^{1/2}}(e_2 - ie_3) \tag{4.23}$$

In this manner we obtain the spinor space S as $C(Q')f = C(Q')y_1y_2$, and the spinor frame

$$\mathscr{S}_0 = \{1 \cdot y_1y_2, x_1 \cdot y_1y_2, x_2 \cdot y_1y_2, x_1x_2 \cdot y_1y_2\} \tag{4.24}$$

Now if we require the existence of half-spinors as the basic quantities, we have to restrict ourselves to the group $\mathrm{Spin}_+(3,1) \subset \mathrm{Pin}(3,1)$. Thus we have the $SO_+(3,1) = \mathscr{L}_0$ as a symmetry group of our original space $R^{3,1}$. The dimension of the spinor space S is equal to $2^r = 4$, and the dimension of the half-spinor spaces S_0 and S_e equals 2. Besides, S_0 is based on vectors

$$S_0 = \{x_1 \cdot y_1y_2, x_2 \cdot y_1y_2\} \tag{4.25}$$

as well as

$$S_e = \{1 \cdot y_1y_2, x_1x_2 \cdot y_1y_2\} \tag{4.26}$$

Now for any orthogonal base ϵ of $R^{3,1}$ the spinor spaces defined by ϵ_0 and ϵ, respectively, are the same, and as the corresponding spinor frames are Pin(3,1) equivalent, when ϵ is obtained from ϵ_0 by means of an operation g, such that $g = \tilde{\varphi}(\gamma)$ for γ satisfying the condition

$$\gamma y_1 y_2 = \pm y_1 y_2 \tag{4.27}$$

that is, for $\gamma \in \mathscr{H} \subset \mathrm{Pin}Q$ and $q \in H = \tilde{\varphi}(\mathscr{H})$. If we shall consider the most interesting case when the spaces of half-spinors are fundamental ones, then we come to the conclusion that the bases ϵ and ϵ_0 must be transformed one into another by means of an element of the connected component of identity H_0 of the group H (Fig. 1). In this case the spinor frames are $\mathrm{Spin}_+ Q \cong SL(2, C)$-equivalent. It can be shown (Crumeyrolle, 1975) that the group $H_0 \subset SO_+(3,1)$ is homeomorphic to R^2.

5. LORENTZ STRUCTURES ON SPACE–TIME

Space–time is usually defined as a smooth, connected, paracompact, Hausdorf four-dimensional manifold M, which carries a smooth global Lorentz tensor field g. These properties of space–time manifold are assumed for a mathematical convenience. It is known that any paracompact, Hausdorf manifold admits a global metric tensor field; from the other side every connected Hausdorf four-dimensional manifold with the Lorentz metric is paracompact (Geroch, 1968). A manifold M admits a Lorentz tensor field if and only if it admits a direction field, and these directions can be chosen to be timelike. Although there is as yet no observational evidence that space–time is orientable, we shall assume the existence of a time orientation of space–time as physically motivated. Then M admits oriented direction field. It is known that there is one-to-one correspondence between the pseudo-Riemannian metrics of signature $(+++-)$ on M and $\mathscr{L}$ structures $\xi_{\mathscr{L}}$ on M (Lichnerowicz, 1968). [We recall that a Lorentz structure on M is a reduction of the structural group $GL(4)$ of the principal frame bundle of M to the Lorentz group $\mathscr{L} = O(3,1)$.] Such a reduction (Whiston, 1973) is possible if and only if the associated bundle $\xi_{GL}[GL(4)/O(3,1)]$ admits a cross section. Let us suppose that M admits a Lorentz structure. It allows us to define the associated bundle $\xi_{\mathscr{L}}[TL]$ of timelike directions with the projective space TL as the fiber. Because TL is contractible (what has been pointed in Section 2), it follows (Husemoller, 1966) that there is a global section of the $\xi_{\mathscr{L}}[TL]$ bundle which is just the timelike direction field on M. Conversely, since TL is homeomorphic to a subspace of $GL(4)/O(1,3)$ (Whiston, 1975), the timelike direction field defines the section of the bundle $\xi_{GL}[GL(4)/O(3,1)]$ that is a reduction of the principal frame bundle over M to the Lorentz group. Because every noncompact four-manifold admits direction field, it admits a Lorentz structure too. Moreover, this field can be oriented, and therefore M is time orientable. The time orientability of the space–time M is equivalent to the possibility of reduction of Lorentz structure $\xi_{\mathscr{L}}$ to the orthochronous Lorentz group $\mathscr{L}^{\uparrow}$, and a space–time is an orientable manifold if and only if a Lorentz structure reduces to the group $SO(3,1) = \mathscr{L}_+$. If M is an orientable and time-orientable space–time manifold, then the principal frame bundle ξ_{GL} can be reduced to the principal

$\mathscr{L}_0$ bundle $\xi_{\mathscr{L}_0}$. There is a connection between the overall orientability of M and the properties of time orientability and space orientability following from the structure of the group $\mathscr{L}$ as a group extension of the appropriate groups (Section 2). Namely, any two of the three possible orientability properties imply the other. Then if M is time orientable and space orientable, then the structure group $\mathscr{L}$ of the $\xi_{\mathscr{L}}$ bundle have to be reduced to $\mathscr{L}_0 = \mathscr{L}_+{}^{\uparrow}$.

6. SPINOR STRUCTURES ON SPACE–TIME

Spinors are used in many physical theories based on flat models of space–time. They are necessary to express very important features of the physical world. In the case of curved space–time we have no physical reasons to give up the possibility of introducing the spinors. On the contrary, an argument of Aharonov and Suskind makes the assumption of the existence of a spinor structure plausible. What do we have to do to introduce spinors in general relativity? First of all we must be able to affix a spin space to every point of space–time M in a continuous way. To do this let us consider the Clifford algebra of the space–time four-manifold M. Because a Lorentz metric g on M defines on the tangent bundle over M a nondegenerate quadratic form of signature (3,1): $Q(X) = g(X, X)$ for $X \in \mathscr{X}(M)$, we shall obtain the Clifford algebra related to Q [which we shall denote $C_M(3,1)$] as the factor algebra of vector fields $\otimes\, \mathscr{X}(M)$ by the ideal generated by the elements $X \otimes X - Q(X)$. The Clifford algebra $C_M(3,1)$ defines in a natural way the Clifford algebra $C_x(3,1)$ at every point of M as the Clifford algebra of tangent space T_xM equipped with the quadratic form Q_x. Now, the spinor space S_x at $x \in M$ will be the minimal left ideal of the algebra $C_x'(3,1)$ [which is the complexification of the Clifford algebra $C_x(3,1)$]. As we have seen, such an ideal can be defined by some orthonormal base $\epsilon_x{}^0$ of T_xM in the following way: The base $\epsilon_x{}^0$ defines the Witt base $\{x_1, x_2, y_1, y_2\}_x$ of the complexification of T_xM and S_x will be equal to $C_x'(3,1)f_x$, with $f_x = y_1 y_2$. In this manner every orthogonal tetrad at $x \in M$ defines some spinor space at the same point x. Now by the local triviality of the Lorentz structure $\xi_{\mathscr{L}}$, there is an open covering $\{U_\alpha\}_{\alpha \in A}$ of the manifold M with the local fields of orthogonal tetrads h_α, where h_α is the local cross section over U_α given by the local trivialization φ_α: $U_\alpha \times \mathscr{L} \to \xi_{\mathscr{L}}|_{U_\alpha}$. This enables us to construct spinor spaces over U_α in a continuous way. But the spinor structure will be uniquely defined if the spinor spaces given by $h_\alpha(x)$ and $h_\beta(x)$ for $x \in U_\alpha \cap U_\beta$ will be the same. We require for physical reasons the existence of half-spinor spaces as fundamental ones. It follows from the considerations of Section 4 that the frame $h_\alpha(x)$ can be obtained from $h_\beta(x)$ by means of an operation belonging to H_0. It means that transition functions of the principal bundle $\xi_{\mathscr{L}}$ must take their values from H_0. However, it is known (Kobayashi, 1963) that this is the necessary and sufficient condition for reducibility of the structure group $\mathscr{L}$ to H_0. Because H_0 is a subgroup of $\mathscr{L}_0$, we obtain that first of all the $\mathscr{L}$ structure $\xi_{\mathscr{L}}$ must be reducible to $\xi_{\mathscr{L}_0}$. To summarize: we can construct the half-spinor spaces over M in a continuous way if and only if the space–time manifold M is time orientable and space orientable, and if there exists an open covering $\{U_\alpha\}$ of M with the set of transition functions which take their values in $H_0 \cong R^2$.

On the other hand, we know that the reduction of the principal bundle of oriented tetrads $\xi_{\mathscr{L}_0}$ to ξ_{H_0} is uniquely defined by a cross section σ of the associated bundle $\xi_{\mathscr{L}_0}[\mathscr{L}_0/H_0]$ (we recall that $\mathscr{L}_0 = SO_3\mathscr{A}\mathscr{N}$). From the earlier consideration it is clear that the cross section σ allows us to define the global field of orthogonal tetrads on M (because the group space of $\mathscr{A}$ and that of $\mathscr{N} \cong R^2$ are contractible). Now let us take the open covering $\{U_\alpha\}_{\alpha\in A}$ of the space–time, with U_α simply connected and with local fields of orthogonal tetrads h_α with $g_{\alpha\beta} \in R^2 \cong H_0$. We shall use two very well known facts. The first is that for every fibering $\pi\colon X \to B$, every mapping of connected and simply connected space U into B can be lifted to the space X:

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ U \arrow[ur, dashed, "g"] \arrow[r, hook, "f"'] & B \end{tikzcd}$$

And the second states that when X is a covering space over B, g as well as g' are two liftings of the given mapping f, and $g(x)$ is equal to $g'(x)$ at least at one point $x \in U$; then $g(x) = g'(x)$ for every $x \in U$. In the considered case $SL(2, C)$ is a covering space over $\mathscr{L}_0$ relative to a map $\tau\colon SL(2, C) \to \mathscr{L}_0$; hence there exists a lifting $\lambda_{\alpha\beta}$ of the map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathscr{L}_0$. From $g_{\alpha\beta} \in H_0 \cong R^2$ we have uniquely defined mappings $\lambda_{\alpha\beta}\colon U_\alpha \cap U_\beta \to SL(2, C)$; their images belong to H_0 too. Then the following relations hold:

$$\lambda_{\alpha\beta}(x) = \lambda_{\alpha\gamma}(x)\lambda_{\gamma\beta}(x) \qquad \text{for } x \in U_\alpha \cap U_\beta \cap U_\gamma$$

It means that $\lambda_{\alpha\beta}$ are transition functions of the principal $SL(2, C)$ bundle $\xi_{SL(2,C)}$. It can be easily checked that there exists the homomorphism S: $\xi_{SL(2,C)} \to \xi_{\mathscr{L}_0}$ such that the following diagram commutes:

$$\begin{tikzcd} B_{SL(2,C)} \times SL(2, C) \arrow[r, "S\times\tau"] \arrow[d] & B_{\mathscr{L}_0} \times \mathscr{L}_0 \arrow[d] \\ B_{SL(2,C)} \arrow[r, "S"] \arrow[dr, "\pi_S"'] & B_{\mathscr{L}_0} \arrow[d, "\pi_0"] \\ & M \end{tikzcd}$$

(where B_G denotes the bundle space of the appropriate G bundle ξ_G). Thus we come to the definition of the spinor structure given by Milnor and Lichnerowicz.

Besides, if space–time is simply connected, then the spinor structure is uniquely defined up to equivalence. Indeed, let $h(x)$ and $h'(x)$ be two different global fields of orthogonal tetrads over M, and φ and φ' two trivializations of $\xi_{\mathscr{L}_0}$ given by h and h', respectively [$\varphi(x, e) = h(x)$ and $\varphi'(x, e) = h'(x)$]. Then we have

$$\begin{tikzcd}
M \times SL(2, C) \arrow[r, "1\times\tau"] \arrow[d] \arrow[rr, bend left, squiggly, "\chi"] & M \times \mathscr{L}_0 \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] & M \times \mathscr{L}_0 \arrow[l, "\varphi'"'] \arrow[d] & M \times SL(2, C) \arrow[l, "1\times\tau"'] \arrow[d] \arrow[ll, bend right, squiggly, "\chi'"'] \\
M \arrow[r, leftrightarrow] & M \arrow[r, leftrightarrow] & M \arrow[r, leftrightarrow] & M \arrow[r, leftrightarrow] & M
\end{tikzcd}$$

where

$$\chi(x, s) = \varphi(x, \tau(s)) = h(x)\tau(s)$$
$$\chi'(x, s) = \varphi'(x, \tau(s)) = h'(x)\tau(s)$$

For simply connected space–time M the map $g\colon M \to \mathscr{L}_0$ defined by $h'(x) = h(x)g(x)$ can be lifted to $\lambda\colon M \to SL(2, C)$, and we obtain

$$\chi'(x, s) = \chi(x, \lambda(x)s)$$

It means that these two spinor structures can be identified.

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